Exercise 2

Use residues to derive the integration formulas in Exercises 1 through 6.

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \int_{-\infty}^\infty \frac{dx}{2(x^2+1)^2}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{1}{2(z^2+1)^2},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$2(z^2+1)^2 = 0$$
$$z^2+1 = 0$$
$$z = \pm i$$

The singular point of interest to us is the one that lies within the closed contour, $z = i$.

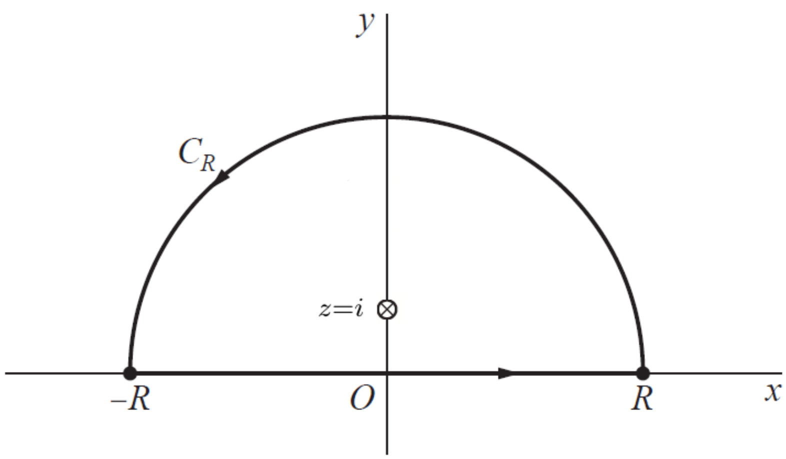

Figure 1: This is Fig. 99 with the singularity at $z = i$ marked.

According to Cauchy's residue theorem, the integral of $1/[2(z^2+1)^2]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{dz}{2(z^2+1)^2} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)^2}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{dz}{2(z^2+1)^2} + \int_{C_R} \frac{dz}{2(z^2+1)^2} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)^2}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, & r = -R \quad &\to \quad r = R \\ C_R: &\quad z = Re^{i\theta}, & \theta = 0 \quad &\to \quad \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{dr}{2(r^2+1)^2} + \int_{C_R} \frac{dz}{2(z^2+1)^2} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)^2}.$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{dr}{2(r^2+1)^2} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)^2}$$

The denominator can be written as $2(z^2+1)^2 = 2(z+i)^2(z-i)^2$. From this we see that the multiplicity of the $z-i$ factor is 2. The residue at $z=i$ can then be calculated by

$$\operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)^2} = \frac{\phi^{(2-1)}(i)}{(2-1)!} = \phi'(i),$$

where $\phi(z)$ is equal to $f(z)$ without $(z-i)^2$.

$$\phi(z) = \frac{1}{2(z+i)^2} \quad \to \quad \phi'(z) = -\frac{1}{(z+i)^3} \quad \Rightarrow \quad \phi'(i) = \frac{1}{8i}$$

So then

$$\operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)^2} = \frac{1}{8i}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dr}{2(r^2+1)^2} &= 2\pi i \left(\frac{1}{8i}\right) \\ &= \frac{\pi}{4}. \end{aligned}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{dz}{2(z^2+1)^2} &= \int_0^\pi \frac{Rie^{i\theta}\,d\theta}{2[(Re^{i\theta})^2+1]^2} \\ &= \int_0^\pi \frac{Rie^{i\theta}\,d\theta}{2(R^2e^{i2\theta}+1)^2}\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{dz}{2(z^2+1)^2}\right| &= \left|\int_0^\pi \frac{Rie^{i\theta}\,d\theta}{2(R^2e^{i2\theta}+1)^2}\right| \\ &\le \int_0^\pi \left|\frac{Rie^{i\theta}}{2(R^2e^{i2\theta}+1)^2}\right| d\theta \\ &= \int_0^\pi \frac{\left|Rie^{i\theta}\right|}{|2(R^2e^{i2\theta}+1)^2|}\,d\theta \\ &= \int_0^\pi \frac{R}{\left|R^2e^{i2\theta}+1\right|^2}\,\frac{d\theta}{2} \\ &\le \int_0^\pi \frac{R}{(|R^2e^{i2\theta}|-|1|)^2}\,\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{(R^2-1)^2}\,\frac{d\theta}{2} \\ &= \frac{\pi}{2}\frac{R}{(R^2-1)^2}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{2(z^2+1)^2}\right| &\le \lim_{R\to\infty} \frac{\pi}{2}\frac{R}{(R^2-1)^2} \\ &= \lim_{R\to\infty} \frac{\pi}{2R^3}\frac{1}{\left(1-\frac{1}{R^2}\right)^2}\end{aligned}$$

The limit on the right side is zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{2(z^2+1)^2}\right| \le 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{2(z^2+1)^2}\,dz\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{dz}{2(z^2+1)^2}\,dz = 0.$$